

Frequency entrainment in long chains of oscillators with random natural frequencies in the weak coupling limit

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We study oscillator chains of the form $\dot{\phi}_k = \omega_k + K[\Gamma(\phi_{k-1} - \phi_k) + \Gamma(\phi_{k+1} - \phi_k)]$, where $\phi_k \in [0, 2\pi)$ is the phase of oscillator k . In the thermodynamic limit where the number of oscillators goes to infinity, for suitable choices of $\Gamma(x)$, we prove that there is a critical coupling strength K_c , above which a stable frequency-entrained state exists, but below which the probability is zero to have such a state. It is assumed that the natural frequencies are random with finite bandwidth. A crucial condition on $\Gamma(x)$ is that it is nonodd, i.e., $|\Gamma(x) + \Gamma(-x)| \neq 0$. The interest in the results comes from the fact that any chain of limit-cycle oscillators can be described by equations of the above form in the limits of weak coupling and narrow distribution of natural frequencies.

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I. INTRODUCTION

Networks of a large number N of coupled limit-cycle oscillators appear in many areas of science [1]. Examples are pacemaker cells in the brain and heart, swarms of fireflies, and applauding audiences. Most often there is a spread in the natural frequencies of the individual oscillators, and the coupling is such that it tends to even out these frequency differences. A natural question to ask is whether there is a critical coupling strength at which a macroscopic number n of oscillators frequency entrain in the thermodynamic limit $N \rightarrow \infty$, and we enter a regime of collective oscillation. In the language of statistical physics, this can be expressed as a phase transition at which the order parameter r becomes nonzero, where

$$r \equiv \lim_{N \rightarrow \infty} n_{\max}/N. \quad (1)$$

Here, n_{\max} is the number of members in the largest connected cluster of entrained oscillators. Such a phase transition is relevant, for example, in the sinus node in the heart. It consists of millions of pacemaker cells, which have to work at a common frequency to trigger regular heartbeats, despite the fact that their natural frequencies differ [2].

We have previously proved that a phase transition of this kind takes place in a one-dimensional chain of pulse-coupled oscillators, interacting like cardiac pacemaker cells [3]. There, the order parameter r jumps discontinuously from zero to one at a critical coupling strength, provided the natural frequencies are random with a sufficiently narrow bandwidth. It would be interesting to obtain a corresponding result for oscillators obeying the equation

$$\dot{\phi}_k = \omega_k + K \sum_{j \in n_k} \Gamma_{jk}(\phi_j - \phi_k), \quad (2)$$

where $\phi_k \in [0, 2\pi)$ is the phase of oscillator k , n_k is the set of oscillators coupled to k , and $\Gamma_{jk}(x)$ are 2π -periodic functions. This is done in the present study. The interest in equations of the form (2) comes from the fact that they describe any network of limit-cycle oscillators in the limits of weak

coupling and narrow distribution of natural frequencies, as shown by Y. Kuramoto [4]. It has also been shown that certain kinds of Josephson junction arrays can be mapped onto models of this kind [5].

Models of the form (2) have been extensively studied. Most often it is assumed that all coupling functions are the same, i.e., that $\Gamma_{jk}(x) = \Gamma(x)$. Kuramoto himself introduced the model $\Gamma(x) = \sin(x)$. He showed that there is a phase transition to a nonzero order parameter r if the coupling is global (all-to-all) [4]. In the case of local (nearest-neighbor) coupling, Strogatz and Mirollo [6] showed that for any finite K , the probability for frequency entrainment ($r=1$) is zero whenever the oscillators are placed on a finite-dimensional lattice, and the natural frequencies are random with nonzero variance. However, it is still an open question as to whether states with $0 < r < 1$ may exist for finite K . The proof of Strogatz and Mirollo rests on the fact that the sine function is odd. Several authors have pointed out that the appearance of frequency entrainment is facilitated by nonodd coupling, i.e., by $\Gamma(x) + \Gamma(-x) \neq 0$. Sakaguchi *et al.* [7] gave a heuristic argument of why a chain of oscillators with $\Gamma(x) = \sin(x - \alpha) + \sin(\alpha)$ and random natural frequencies may frequency entrain if $\alpha \neq 0$. Kopell and Ermentrout [8] showed that a nonodd coupling enables frequency entrainment in a chain where the natural frequencies obey $|\omega_{k+1} - \omega_k| = O(1/N)$. However, this smoothness condition does not allow random natural frequencies. Rogers and Wille [9] considered the Kuramoto oscillator chain $\Gamma(x) = \sin(x)$ with random natural frequencies when $K \propto y^{-\alpha}$. Here, y is the distance between two oscillators. They found numerically that states with $r = 1$ are possible for finite K when $\alpha < 2$. This is consistent with the analytical results that showed they are possible with global coupling ($\alpha=0$), but impossible with local coupling ($\alpha=\infty$).

II. ANALYSIS

In the present paper, we consider an oscillator chain

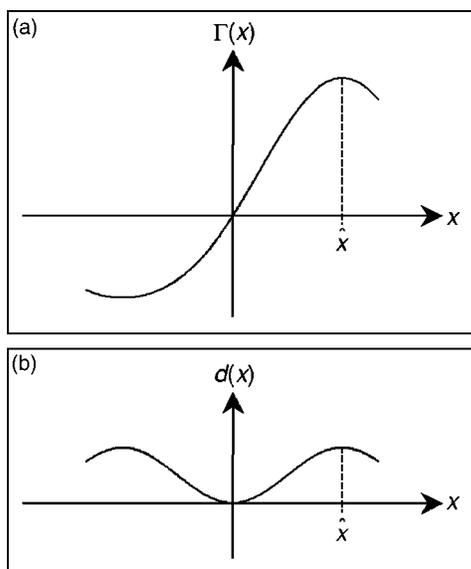


FIG. 1. (a) We look for entrained states fulfilling Eq. (5), where \hat{x} is defined as the largest x for which $\Gamma'(x) > 0$ whenever $|x| < \hat{x}$. (b) The asymmetry function $d(x) = \Gamma(x) + \Gamma(-x)$. In both panels, the coupling function (30) is used, with $a = 0.25$.

$$\dot{\phi}_k = \omega_k + K[\Gamma(\phi_{k-1} - \phi_k) + \Gamma(\phi_{k+1} - \phi_k)]. \quad (3)$$

We use open boundary conditions, i.e., oscillator 1 is coupled only to oscillator 2, and N is coupled only to $N-1$. We assume that $\Gamma(x)$ is a continuous function, which is at least piecewise smooth. Furthermore, we assume that

$$\begin{cases} \Gamma(0) = 0 \\ \Gamma'(0^+) \geq 0, \quad \Gamma'(0^-) \geq 0. \end{cases} \quad (4)$$

This corresponds to diffusive coupling, which tends to even out phase differences if they are small enough. We look for frequency-entrained states where $\dot{\phi}_k(t) = \Omega$ for all k . The constant Ω is the entrained frequency. In such states, the $N-1$ phase differences $\Delta\phi_k \equiv \phi_{k+1} - \phi_k$ are all constants. We restrict our interest to states for which

$$|\Delta\phi_k| < \hat{x}, \quad \forall k, \quad (5)$$

where \hat{x} is defined in Fig. 1(a).

In an entrained state we may write

$$\Omega = \omega_k + K[\Gamma(-\Delta\phi_{k-1}) + \Gamma(\Delta\phi_k)]. \quad (6)$$

Let us introduce the vectors $\omega = (\omega_1, \dots, \omega_N)$ and $\Delta\phi = (\Delta\phi_1, \dots, \Delta\phi_{N-1})$. Rearranging terms in Eq. (6), we may then express a function $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$

$$\omega = f(\Omega, \Delta\phi). \quad (7)$$

This function is invertible in the subspace $\{\Omega, \Delta\phi; |\Delta\phi_k| < \hat{x}, \forall k\}$, in which we look for entrained states [10], so that we may write

$$(\Omega, \Delta\phi) = f^{-1}(\omega). \quad (8)$$

In other words, there is at most one entrained state $(\Omega, \Delta\phi)$ of the desired kind (5) for a given assignment of natural frequencies.

As expected from previous work, it turns out that the nonoddity, or asymmetry, of $\Gamma(x)$ is crucial for the appearance of frequency entrainment. We define the even ‘‘asymmetry function’’ $d(x)$ as

$$d(x) = \Gamma(x) + \Gamma(-x) \quad (9)$$

[Fig. 1(b)]. For the Kuramoto model where $\Gamma(x) = \sin(x)$, we have $d(x) \equiv 0$. For simplicity, we restrict our interest to coupling functions $\Gamma(x)$, for which $d(x)$ is either monotonically increasing, or monotonically decreasing in the interval $0 \leq x \leq \hat{x}$. (See Sec. IV for a discussion of this point.) Thus we consider the two cases

$$\begin{cases} d(0) = 0 \\ d'(x) \geq 0, \quad 0 \leq x \leq \hat{x}. \end{cases} \quad (10)$$

[Fig. 1(b)], and

$$\begin{cases} d(0) = 0 \\ d'(x) \leq 0, \quad 0 \leq x \leq \hat{x}. \end{cases} \quad (11)$$

We first prove the following:

Proposition 1. For any N , an entrained state of the desired kind (5) exists whenever $K > K_c$, where

$$K_c = \frac{\omega_{\max} - \omega_{\min}}{|d(\hat{x})|}. \quad (12)$$

Here, ω_{\max} and ω_{\min} are the maximum and minimum natural frequencies, respectively.

Our next result is:

Proposition 2. Let ω_k be independent random numbers from a distribution with support $[\omega_{\min}, \omega_{\max}]$. Then, in the limit $N \rightarrow \infty$, if an entrained state of the desired kind (5) exists, the probability is one that $\Omega = \omega_{\max}$ in case (10), and that $\Omega = \omega_{\min}$ in case (11). Also, the probability is zero that the system has such an entrained state when $K < K_c$.

The last statement is the most important one. Together with proposition 1, it implies that for random natural frequencies, K_c becomes a well-defined critical coupling in the thermodynamic limit, in the sense that the probability is zero to have an entrained state of the desired kind (5) when $K < K_c$, and that it is 1 when $K > K_c$. For the Kuramoto model, we should enter zero in the denominator of Eq. (12), meaning that there is no finite coupling strength at which the chain frequency entrains in the thermodynamic limit, whenever the distribution of natural frequencies has nonzero bandwidth. This agrees with the results by Strogatz and Mirollo [6] discussed in Sec. I.

Finally, we prove:

Proposition 3. The desired entrained states (5) are locally stable.

This ensures that the phase transition at K_c can actually be seen. Now we go on to prove the three statements. We proceed along the same lines as in Ref. [3], but the arguments become clearer for the present model. We only consider ex-

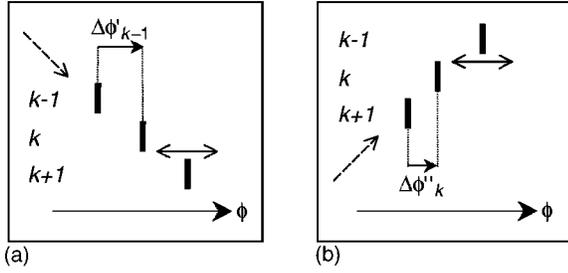


FIG. 2. (a) The state $(\omega_{\max}, \Delta\phi')$ is constructed from the top to the bottom of the chain by adjusting $\Delta\phi'_k$ so that oscillator k is accelerated to frequency ω_{\max} , given the decelerating influence it receives from $k-1$. The bars represent the phases of the oscillators at a given time. (b) The state $(\omega_{\max}, \Delta\phi'')$ is constructed from bottom to top by adjusting $\Delta\phi''_{k-1}$ so that oscillator k is accelerated to frequency ω_{\max} . Compare to Fig. 3.

explicitly the case (10) where $d(x) \geq 0$. At the end of Sec. II we comment on case (11), where $d(x) \leq 0$.

Proof of Proposition 1. Let us try to entrain all oscillators to $\Omega = \omega_{\max}$ in the following way: Since $\Omega \geq \omega_k$ for all k , no oscillator shall be decelerated. We start by adjusting $\Delta\phi'_1 \geq 0$ so that oscillator 1 is accelerated to ω_{\max} , then we adjust $\Delta\phi'_2 \geq 0$ so that oscillator 2 is accelerated to ω_{\max} , given the decelerating influence it receives from 1, due to the already adjusted $\Delta\phi'_1 \geq 0$. Then, we continue in this way down the chain, as illustrated in Fig. 2(a).

The situation when it is the most difficult to entrain an oscillator k to ω_{\max} in this way is when $\omega_k = \omega_{\min}$ and $\Delta\phi'_{k-1} = \hat{x}$. Then, we have

$$\omega_{\max} = \omega_{\min} + K[\Gamma(-\hat{x}) + \Gamma(\Delta\phi_k)] \quad (13)$$

from Eq. (6). It follows from the assumptions on $\Gamma(x)$ that this equation is fulfilled for some $\Delta\phi$ with $|\Delta\phi_k| < \hat{x}$ for all $K > K_c$ [Eq. (12)], but not for any $K \leq K_c$. Therefore, if $K > K_c$, we can always construct an entrained state of the desired kind for all oscillators—except the last one. Oscillator N is decelerated by $N-1$, but there is no oscillator $N+1$ with which we can speed it up to frequency ω_{\max} . However, we may say that we have succeeded in finding the entrained state

$$(\omega_{\max}, \Delta\phi') = f^{-1}(\omega'), \quad (14)$$

for the slightly different assignment of natural frequencies $\omega' = (\omega_1, \dots, \omega_{N-1}, \omega'_N)$, where $0 \leq \Delta\phi'_k < \hat{x}$, $\forall k$, and

$$\omega'_N = \omega_{\max} - K\Gamma(-\Delta\phi'_{N-1}) \geq \omega_{\max}. \quad (15)$$

By starting at the bottom of the chain [Fig. 2(b)] with oscillator N , we can, in the same way, construct an entrained state

$$(\omega_{\max}, \Delta\phi'') = f^{-1}(\omega'') \quad (16)$$

for $\omega'' = (\omega''_1, \omega_2, \dots, \omega_N)$, where $-\hat{x} < \Delta\phi''_k \leq 0$, $\forall k$, and

$$\omega''_1 = \omega_{\max} - K\Gamma(\Delta\phi''_1) \geq \omega_{\max}. \quad (17)$$

Now we want to show that an entrained state $(\Omega, \Delta\phi) = f^{-1}(\omega)$ exists for some $\Delta\phi$ with

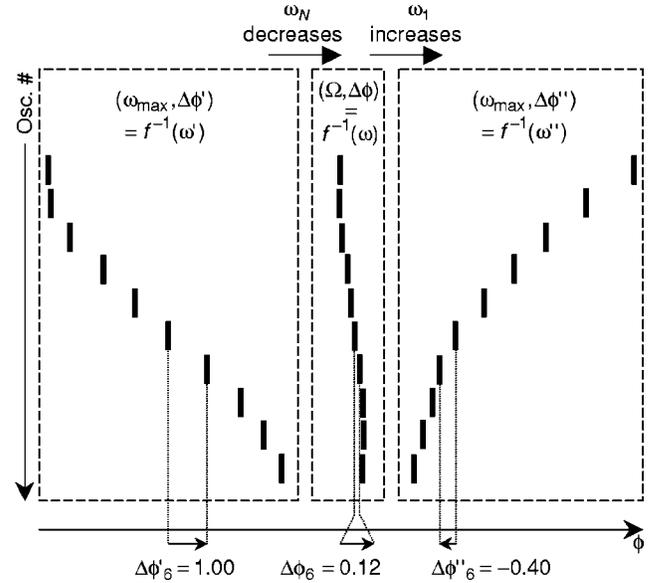


FIG. 3. The idea of how to prove proposition 1. Phase profiles of chains of ten frequency-entrained oscillators. The left entrained state $(\omega_{\max}, \Delta\phi')$ is constructed from top to bottom, according to Fig. 2(a). The right state $(\omega_{\max}, \Delta\phi'')$ is constructed from bottom to top, according to Fig. 2(b). The true entrained state $(\Omega, \Delta\phi)$ is found between these states in the sense that $\Delta\phi'_k \geq \Delta\phi_k \geq \Delta\phi''_k$. See the text for further explanation. In the figure, entrained states of the model (30) are shown, with $\alpha=0.5$ and $K=1.0$. The natural frequencies $\omega = (1.44, 1.03, 1.04, 1.24, 1.23, 1.08, 1.26, 1.49, 1.50, 1.25)$ are random numbers from a square distribution with $\omega_{\min} = 1.0$ and $\omega_{\max} = 1.5$. This gives $K_c = 1.0$ according to Eq. (12).

$$\Delta\phi'_k \geq \Delta\phi_k \geq \Delta\phi''_k, \quad \forall k \quad (18)$$

(Fig. 3). It then follows that $-\hat{x} < \Delta\phi_k < \hat{x}$, as requested. We succeed in doing this if we can show that

$$\frac{\partial \Delta\phi_k}{\partial \omega_1} < 0, \quad \frac{\partial \Delta\phi_k}{\partial \omega_N} > 0, \quad \forall k \quad (19)$$

for an entrained state $(\Omega, \Delta\phi) = f^{-1}(\omega)$ of the desired kind. Then $\Delta\phi_k$ will decrease throughout the process

$$(\omega_{\max}, \Delta\phi') \xrightarrow{\omega_N \text{ decreases}} (\Omega, \Delta\phi) \xrightarrow{\omega_1 \text{ increases}} (\omega_{\max}, \Delta\phi''), \quad (20)$$

and $\Delta\phi_k$ is trapped in the allowed range, as expressed in Eq. (18). To prove Eq. (19), we use Eq. (6) to write

$$\begin{cases} \Gamma(\Delta\phi_1) = K^{-1}(\Omega - \omega_1) \\ \vdots \\ \Gamma(\Delta\phi_k) = K^{-1}(\Omega - \omega_k) - \Gamma(-\Delta\phi_{k-1}) \\ \vdots \\ \Gamma(\Delta\phi_{N-1}) = K^{-1}(\Omega - \omega_{N-1}) - \Gamma(-\Delta\phi_{N-2}). \end{cases} \quad (21)$$

But we can also write

$$\Gamma(-\Delta\phi_{N-1}) = K^{-1}(\Omega - \omega_N). \quad (22)$$

If we differentiate the left- and right-hand sides of Eq. (21)

with respect to ω_1 , we get equations of the form

$$\begin{cases} \frac{\partial \Delta \phi_1}{\partial \omega_1} = c_{11} \frac{\partial \Omega}{\partial \omega_1} - c_{12} \\ \vdots \\ \frac{\partial \Delta \phi_k}{\partial \omega_1} = c_{k1} \frac{\partial \Omega}{\partial \omega_1} + c_{k2} \frac{\partial \Delta \phi_{k-1}}{\partial \omega_1} \\ \vdots \\ \frac{\partial \Delta \phi_{N-1}}{\partial \omega_1} = c_{N-1,1} \frac{\partial \Omega}{\partial \omega_1} + c_{N-1,2} \frac{\partial \Delta \phi_{N-2}}{\partial \omega_1}. \end{cases} \quad (23)$$

We have $c_{jk} > 0$ for all j, k , since $\Gamma'(x) > 0$ for all $\Delta \phi_k$ of interest. But if we differentiate Eq. (22) in the same way, we get

$$\frac{\partial \Delta \phi_{N-1}}{\partial \omega_1} = -c_{N,1} \frac{\partial \Omega}{\partial \omega_1}, \quad (24)$$

with $c_{N,1} > 0$. We proceed to use Eqs. (23) and (24) to show $\partial \Delta \phi_k / \partial \omega_1 < 0$ by contradiction. Suppose that $\partial \Omega / \partial \omega_1 \leq 0$. Then $\partial \Delta \phi_1 / \partial \omega_1 < 0$ from the first row of Eq. (23), $\partial \Delta \phi_2 / \partial \omega_1 < 0$ from the second row, and so on. From the last row we get $\partial \Delta \phi_{N-1} / \partial \omega_1 < 0$, while we get $\partial \Delta \phi_{N-1} / \partial \omega_1 \geq 0$ from Eq. (24). We have a contradiction. Thus,

$$\partial \Omega / \partial \omega_1 > 0. \quad (25)$$

From Eq. (24) it follows that $\partial \Delta \phi_{N-1} / \partial \omega_1 < 0$. From the last row of Eq. (23), it then follows that $\partial \Delta \phi_{N-2} / \partial \omega_1 < 0$, from the second to last row that $\partial \Delta \phi_{N-3} / \partial \omega_1 < 0$, and so on, so that $\partial \Delta \phi_k / \partial \omega_1 < 0$ for all k . That

$$\partial \Omega / \partial \omega_N > 0, \quad (26)$$

and $\partial \Delta \phi_k / \partial \omega_N > 0$ for all k can be shown in an analogous way, or seen by symmetry. ■

Proof of Proposition 2. We first show that $\Omega \rightarrow \omega_{\max}$ as $N \rightarrow \infty$. Applying Eqs. (25) and (26) to the process (20), we see that $\Omega \leq \omega_{\max}$. Suppose that $\Omega < \omega_{\max}$. In the limit $N \rightarrow \infty$, there is then, with probability, one chain segment $\{j, j+1, \dots, j+M\}$ in which all natural frequencies ω_k fulfil $\omega_k > L$ for some L with $\omega_{\max} > L > \Omega$ (Fig. 4). This holds for all fixed M , arbitrarily large. All oscillators in this segment should then be decelerated. We want to show that this implies that $|\Delta \phi_k|$ grows without bound in the segment, so that such entrained states cannot exist. From Eq. (6), we have

$$\Gamma(\Delta \phi_k) = -\Gamma(-\Delta \phi_{k-1}) - K^{-1}(\omega_k - \Omega) < -\Gamma(-\Delta \phi_{k-1}) - C,$$

with $C \equiv K^{-1}(L - \Omega) > 0$. Since $\Gamma(-\Delta \phi_{k-1}) = d(\Delta \phi_{k-1}) - \Gamma(\Delta \phi_{k-1})$ from Eq. (9), and $d(\Delta \phi_{k-1}) \geq 0$ by assumption, we have $\Gamma(\Delta \phi_k) < \Gamma(\Delta \phi_{k-1}) - C$ and

$$\Gamma(\Delta \phi_{j+M}) < \Gamma(\Delta \phi_j) - MC.$$

Therefore, if M is chosen large enough, $\Gamma(\Delta \phi_{j+M})$ becomes less than the minimum value of $\Gamma(x)$. Therefore, we must have $\Omega = \omega_{\max}$ as we claimed.

We proceed to show that the probability is zero to have an entrained state of the desired kind when $K < K_c$. In the same way as above, the probability is 1 that there is a chain segment $\{\tilde{j}, \tilde{j}+1, \dots, \tilde{j}+\tilde{M}\}$, in which all oscillators k have

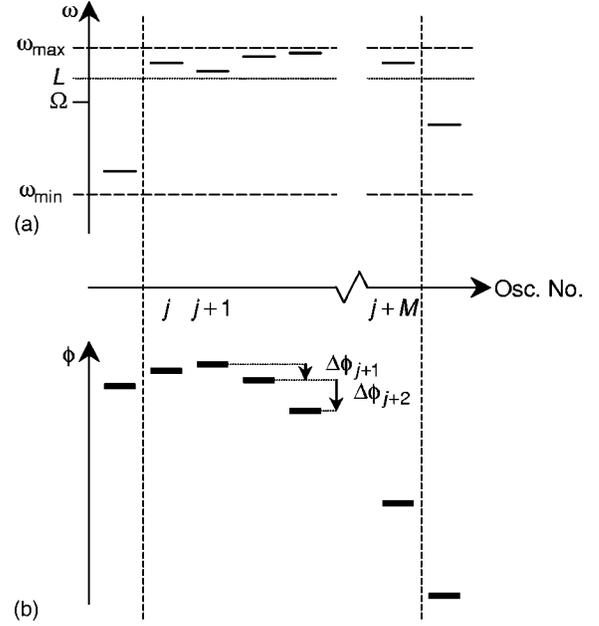


FIG. 4. (a) If we assume random natural frequencies in the limit $N \rightarrow \infty$, there will always be long segments $\{j, \dots, j+M\}$ of the oscillator chain, in which all oscillators have a natural frequency $\omega_k > \Omega$, whenever the entrained frequency $\Omega < \omega_{\max}$. (b) When we try to decelerate all these oscillators to frequency Ω , the phase differences $\Delta \phi_k$ inevitably get smaller and smaller as we move to the right, so that the condition $\Delta \phi_k > -\hat{x}$ is finally broken. From this follows that $\Omega = \omega_{\max}$ by contradiction.

$\omega_k < \omega_{\min} + \delta$, for any \tilde{M} and any positive δ , however small [Fig. 5(a)]. From Eq. (6) and the fact that $\Omega = \omega_{\max}$, we then have

$$\omega_{\max} < \omega_{\min} + \delta + K[\Gamma(-\Delta \phi_{k-1}) + \Gamma(\Delta \phi_k)].$$

Rearranging terms and using Eqs. (9) and (12),

$$\Gamma(\Delta \phi_k) > \Gamma(\Delta \phi_{k-1}) + K^{-1}[K_c d(\hat{x}) - K d(\Delta \phi_{k-1}) - \delta]. \quad (27)$$

We have $\Delta \phi_{k-1} < \hat{x}$ in an entrained state of the desired kind, and using the assumptions in case (10), we get $d(\hat{x}) \geq d(\Delta \phi_{k-1})$, so that

$$\Gamma(\Delta \phi_k) > \Gamma(\Delta \phi_{k-1}) + K^{-1}[(K_c - K)d(\hat{x}) - \delta]. \quad (28)$$

Thus, if $(K_c - K)d(\hat{x}) > \delta$ for any positive δ , then

$$\Gamma(\Delta \phi_{j+M}) > \Gamma(\Delta \phi_j) + \tilde{M} \tilde{C} \quad (29)$$

for some positive $\tilde{C} \equiv K^{-1}[(K_c - K)d(\hat{x}) - \delta]$. Therefore, $\Gamma(\Delta \phi_{j+M})$ becomes larger than the maximum value of $\Gamma(x)$ if \tilde{M} is chosen large enough. Since this is impossible, and we can choose δ as small as we like, we must have $K \geq K_c$. ■

Proof of Proposition 3. We investigate the evolution of a trajectory $\phi'(t) = \phi(t) + \delta \phi(t)$, where $\phi(t)$ is the periodic trajectory corresponding to the entrained state. Linearizing the system (3), we get $\delta \phi = KJ \delta \phi$, with

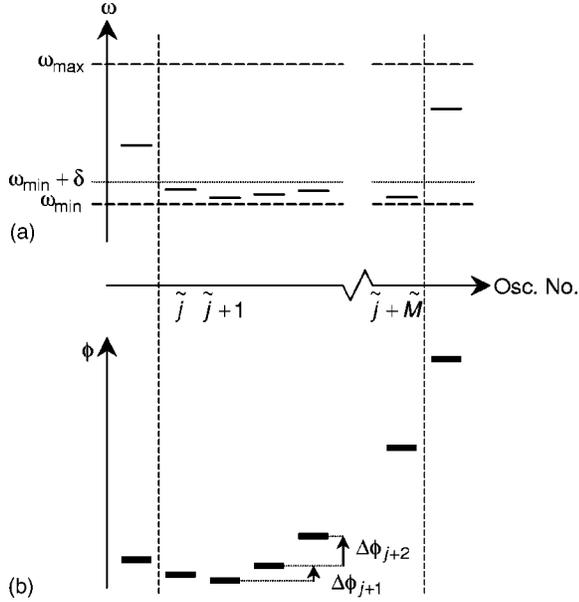


FIG. 5. (a) If we assume random natural frequencies in the limit $N \rightarrow \infty$, there will be arbitrarily long segments $\{\tilde{j}, \dots, \tilde{j} + \tilde{M}\}$ of the oscillator chain, in which all oscillators have a natural frequency ω_k close to the minimum frequency ω_{\min} . (b) If $K < K_c$ and we try to accelerate all these to frequency $\Omega = \omega_{\max}$, the phase differences $\Delta\phi_k$ will grow more and more as we move to the right, so that the condition $\Delta\phi_k < \hat{x}$ is finally broken. Therefore, the chain can only frequency entrain if $K \geq K_c$.

$$J = \begin{pmatrix} -b_1 & b_1 & & & 0 \\ a_2 & -(a_2 + b_2) & b_2 & & \\ & a_3 & -(a_3 + b_3) & b_3 & \\ & & & \ddots & \\ 0 & & & & a_N - a_N \end{pmatrix}.$$

Here, $a_k = \Gamma'(-\Delta\phi_{k-1})$ and $b_k = \Gamma'(\Delta\phi_k)$. From the assumptions (4) and (5), it follows that all a_k and b_k are non-negative. $\delta\phi = \alpha(1, 1, \dots, 1)$ is an eigenvector to J with eigenvalue $\lambda = 0$, and corresponds to a perturbation along $\phi(t)$. This entrained trajectory is stable if the eigenvalue $\lambda = 0$ is nondegenerate, and all other eigenvalues have a negative real part. We see that $\lambda = 0$ is nondegenerate since the equation $0 = J\delta\phi$ implies $\delta\phi_2 = \delta\phi_1$ from the first row, $\delta\phi_3 = \delta\phi_2$ from the second, and so on, so that $\delta\phi = \alpha(1, 1, \dots, 1)$ is the only possible eigenvector.

It follows from Gershgorin's theorem that each eigenvalue λ must fulfil

$$|\lambda - J_{jj}| \leq \sum_{k:k \neq j} |J_{jk}|$$

for some j . Therefore, we have

$$|\lambda + (a_j + b_j)| \leq a_j + b_j,$$

for some j . (To make this equation valid for all j , we define $a_1 = 0$ and $b_N = 0$.) Since $a_j + b_j$ is a non-negative real number for all j , it follows that $\text{Re}[\lambda] \leq 0$, and we are done. ■

Let us finally comment on the case (10) where $d(x) \leq 0$, without going into details. To prove proposition 1, we try to entrain all oscillators to $\Omega = \omega_{\min}$, starting first from the top, giving a state with all $\Delta\phi'_k \leq 0$, then from the bottom, giving $\Delta\phi''_k \geq 0$. The true entrained state is then trapped between these, just like before. To prove the first part of proposition 2, we see that $\Omega > \omega_{\min}$ gives rise to a long segment of oscillators that should all be accelerated. Then we show that this means that $\Delta\phi_k$ grows without bound in the segment, so that we must have $\Omega = \omega_{\min}$. To prove the second part of proposition 2, we use a long chain segment where all oscillators have $\omega_k > \omega_{\max} - \delta$, and then show that $K < K_c$ implies that $\Delta\phi_k$ drops without bound in the segment, so that we must have $K \geq K_c$.

III. SIMULATIONS

We simulated long oscillator chains, to compare the resulting data with the analytical results obtained in Sec. II. The natural frequencies were taken from a square distribution with $\omega_{\min} = 2\pi$ t.u.⁻¹ and $\omega_{\max} = 3\pi$ t.u.⁻¹. The forward Euler integration method was used, with time step $dt = 0.05$ t.u. To check the accuracy of the integration, some simulations with $dt = 0.01$ t.u. were performed. The differences were found to be negligible. Figure 6 shows results from simulations of chains with $N = 20\,000$ oscillators, using the coupling function

$$\Gamma(x) = \sin(x) + a \sin^2(x). \quad (30)$$

A transient time of 100 000 t.u. was allowed before the mean frequency of each oscillator was measured during 1000 t.u. The long transient time was necessary since the standard deviation σ of the distribution of the mean frequencies in the chain converged only after such a long time around the coupling strength K at which the entrainment settled. The error bars show the standard deviation of data from seven independent realizations (assignments of natural frequencies). The initial condition was always $\phi_k(0) = 0$ for all k . The attained mean frequencies in the chain stayed the same when different initial conditions were tested in a given realization. This indicates that the system only has one attractor, and that the presented results are independent of the initial condition. We cannot prove this, however.

For $a = 0.5$, frequency entrainment settled around $K = 0.45$ [Fig. 6(a)], to be compared with the analytical result $K_c = 0.5$. For $a = 0.25$, the corresponding figures are $K \approx 0.8$ and $K_c = 1.0$, respectively. For $a = 0$ (the Kuramoto model), the chain did not frequency entrain in the range of the investigated values of K , in accordance with the result $K_c = \infty$. Figure 6(b) shows the correlation length ξ for the three models as functions of K . The quantity ξ was defined as follows: Let $\bar{\omega}(x)$ be the attained mean frequency of the oscillator at position x (where $x = 1, 2, 3, \dots$). Then the correlation function is $\Gamma(y) = \bar{\Gamma}(y) / \bar{\Gamma}(0)$, where

$$\bar{\Gamma}(y) = \langle [\bar{\omega}(x+y) - \langle \bar{\omega} \rangle][\bar{\omega}(x) - \langle \bar{\omega} \rangle] \rangle_x,$$

and ξ was chosen as

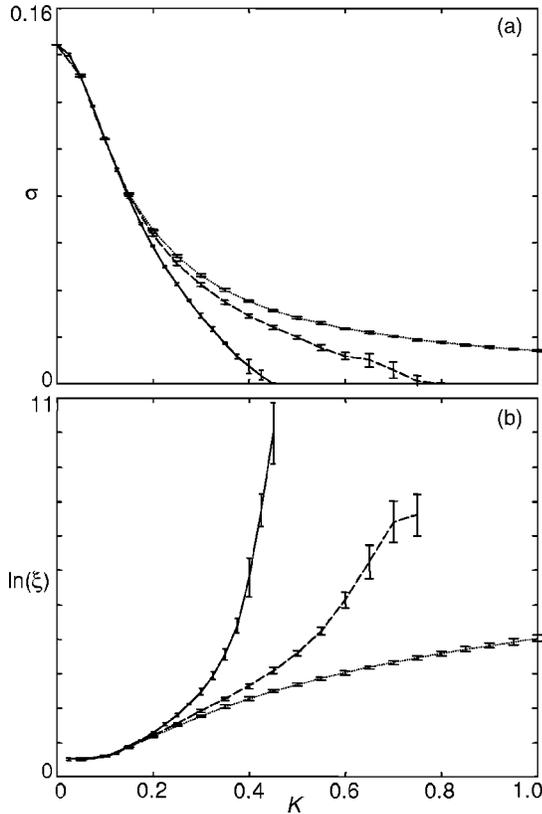


FIG. 6. The results from simulations where the coupling function (30) was used. $a=0.5$ (solid), 0.25 (dashed), and 0 (dotted). (a) The standard deviation of the distribution of the mean frequencies as functions of K . (b) The (logarithm of) correlation length ξ among the mean frequencies in the chain as functions of the coupling strength K . For $N \rightarrow \infty$, we analytically have that $\sigma \rightarrow 0$ and $\xi \rightarrow \infty$ at the critical couplings $K_c=0.5, 1.0$, and ∞ for $a=0.5, 0.25$, and 0 , respectively.

$$\xi = y, \quad \Gamma(y)/\Gamma(1) = 1/e.$$

In the cases where $a=0.5$ and 0.25 , $\xi(K)$ grew faster than the exponential, and could be well fitted to a power-law divergence

$$\xi \propto (\hat{K}_c - K)^\alpha \quad (31)$$

(not shown). The best fits were obtained with \hat{K}_c close to the values at which $\sigma \rightarrow 0$ in Fig. 6(a), but the fits were within the error margins also with $\hat{K}_c = K_c$. In contrast, for the Kuramoto model $a=0$, the growth of $\xi(K)$ was slower than the exponential, and therefore ξ cannot diverge at a finite coupling. Again, this agrees with the theoretical prediction.

Apparently, even in the long chains we used, frequency entrainment settled at a coupling strength substantially smaller than K_c . In Sec. IV we argue that this should always be the case if $\Gamma(x)=0$ only at $x=0$. To investigate the behavior of a “completely asymmetric” model, where $\Gamma(x)=0$ whenever $x \leq 0$, we simulated the model with coupling function

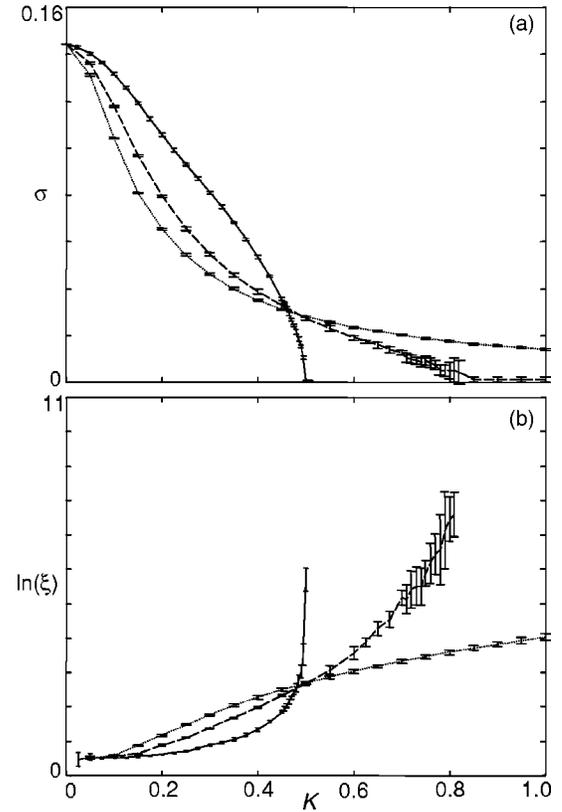


FIG. 7. The results from simulations where the coupling function (32) was used. $a=0$ (solid), $b=0.5$ (dashed), $b=1.0$ (dotted). (a) The standard deviation of the distribution of the mean frequencies as functions of K . (b) The (logarithm of) correlation length ξ among the mean frequencies in the chain as functions of the coupling strength K . For $N \rightarrow \infty$, we analytically have that $\sigma \rightarrow 0$ and $\xi \rightarrow \infty$ at the critical couplings $K_c=0.5, 1.0$, and ∞ for $a=0, 0.5$, and 1.0 , respectively.

$$\Gamma(x) = c(x)\sin(x), \quad c(x) = \begin{cases} b, & x \leq 0 \\ 1, & x > 0 \end{cases} \quad (32)$$

for $b=0.5$ and 0 . (Again, $b=1$ is the Kuramoto model.) The results are shown in Fig. 7. For $b=0.5$, frequency entrainment settled at $K \approx 0.85$, again significantly smaller than $K_c = 1$. We note that frequency entrainment was actually present at the four last points along the dashed curve in Fig. 7(a). The reason why $\sigma > 0$ is that the transient time was too short. For $b=0$, the chain frequency entrained more or less exactly at $K_c=0.5$. Also, the correlation length $\xi(K)$ [Fig. 7(b)] was very well fitted to the power-law divergence (31), with $\hat{K}_c = K_c$ (not shown). Such a fitting was also possible for $b=0.5$, with \hat{K}_c larger than 0.85 .

IV. DISCUSSION

Why does frequency entrainment settle well below K_c in most cases? We think it is because the probability is very low of having the long segments of the chain with high natural frequencies (Fig. 4) used to prove that $\Omega = \omega_{\max}$, and the long segments with low natural frequencies (Fig. 5), which are

then needed to prove that frequency entrainment cannot settle for $K < K_c$. In other words, we think that, as N grows, the probability $P(N, K)$ to have frequency entrainment very slowly approaches the step function that jumps from zero to one at K_c . It might be possible to compute the function $P(N, K)$, but we have not attempted such a derivation. The situation is different for the “completely asymmetric” coupling functions where an oscillator speeds up a phase-delayed neighbor, but is not itself decelerated by this neighbor [e.g., model (32) with $b=0$]. In this case, Ω is always the frequency of the fastest oscillator, which will be very close to ω_{\max} if N is reasonably large. It is then enough to have *one* slow oscillator with a natural frequency very close to ω_{\min} to rule out frequency entrainment for $K < K_c$. Again, such oscillators will be present if N is reasonably large. Consequently, the frequency entrainment settled very close to K_c for model (32) with $b=0$ in Fig. 7.

Let us try to explain these statements. If the model is completely asymmetric in the above sense, we can construct the entrained state by hand as follows: Identify the oscillator k with the highest natural frequency $\omega_k \approx \omega_{\max}$. Then, phase delay its upper and lower neighbors by the appropriate amounts, so that they are also accelerated to frequency ω_k . By assumption, the frequency of oscillator k is not affected by this. Continue in the same way upwards and downwards in the chain, and phase delay each new oscillator, until the top and bottom are reached. Whenever we come across a slow oscillator j with $\omega_j \approx \omega_{\min}$, a phase delay with the maximum allowed magnitude \hat{x} is not enough if $K < K_c$.

It also seems that the more asymmetric the coupling function, the closer to K_c does frequency entrainment settle—if we dare to judge from just the two cases $a=0.25$ and 0.5 in Fig. 6. This would be consistent with the considerations above. We note that these findings are the same as in Ref. [3], where a chain of pulse-coupled oscillators frequency entrained slightly below the critical coupling for a symmetric phase response curve (where phase delays were as large as phase advances), but just at the critical coupling for a completely asymmetric phase response curve (where only phase advances were possible).

The models we used are of the form (2), generally valid in the limits of weak coupling and narrow distribution of natural frequencies. However, we made the restrictive assumptions that all coupling functions were the same, i.e., $\Gamma_{jk}(x) = \Gamma(x)$, and that the coupling was diffusive [Eq. (4)]. In addition, we limited ourselves to coupling functions $\Gamma(x)$ fulfilling either case (10) or (11). The last limitation is not essential if we just want to make sure that there is a critical coupling. For any $\Gamma(x)$ fulfilling Eq. (4), there will always be a $\tilde{x} \leq \hat{x}$ such that $\Gamma(x)$ fulfils either Eq. (10) or (11) if we replace \hat{x} with \tilde{x} . Then there will always be an entrained state

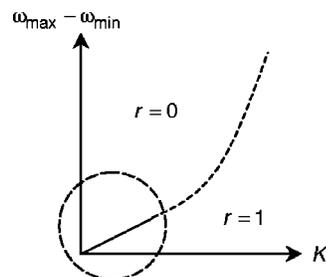


FIG. 8. The phase diagram of an oscillator chain. In the phase with order parameter $r=0$, there are only microscopic clusters of entrained oscillators, whereas all oscillators are entrained when $r = 1$. We study the critical line separating the two phases close to the origin, where dynamical equations of the form (2) can be used.

with $|\Delta\phi_k| < \tilde{x}$ if $K > (\omega_{\max} - \omega_{\min}) / |d(\tilde{x})|$, as shown in exactly the same way as proposition 1.

Thus, we have shown that a quite arbitrary chain of limit-cycle oscillators possesses two phases with order parameter $r=0$ and 1 , respectively, close to the origin of a phase diagram such as that in Fig. 8. It is natural to assume that the critical line separating these two phases can be extrapolated to higher coupling and wider distribution of natural frequencies, provided it exists close to the origin. In other words, our results suggest that a general chain of limit-cycle oscillators with inherently nonodd diffusive coupling possesses these two phases. Phases with $0 < r < 1$ are forbidden in one-dimensional oscillator lattices. If $0 < r < 1$, there should be a nonzero density of oscillators that are not entrained to a presumed infinite cluster. However, these oscillators necessarily break the infinite cluster into finite parts, so that we are left with a state where $r=0$.

Most often the appearance of order is facilitated when the network connectivity is increased. Therefore, we expect that perfect frequency entrainment ($r=1$) also appears in d -dimensional lattices where $d \geq 2$, provided the coupling is nonodd and the bandwidth of the natural frequencies is finite. However, it is not clear to us how to prove this. The method used in this paper cannot be directly generalized to higher dimensions. When $d \geq 2$, states with $0 < r < 1$ are possible and may appear when the natural frequency distribution has tails or the coupling is not strong enough to allow perfect entrainment. There is numerical evidence for such states [11], but analytical results are again lacking.

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- [10] Write $v=f(u)$. If f is to map two points u_1 and u_2 to the same v , then, for some j and k , $\partial v_j/\partial u_k$ must change sign at some u . But $\partial\omega_j/\partial\Omega>0$, and $\partial\omega_j/\partial\Delta\phi_k$ is either identically zero, or strictly positive or negative when $|\Delta\phi_k|<\hat{x}$.
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